$$\alpha = r_0\delta + \nu, \quad \gamma = \mu(r_0\delta + \nu), \; -\beta = \mu\delta + {}^1/_2 a$$

Then conditions (2.5) have the form

$$\mu (r_0\delta + \nu)^2 - (\mu\delta + {}^1/_2 a)^2 > 0, \qquad \mu (r_0\delta' + \nu')^2 - (\mu\delta' + {}^1/_2 a')^2 > 0 \qquad (2.6)$$

The motion (0.2) of a body is hence stable.

BIBLIOGRAPHY

1. Sretenskii, L. N., On some cases of motion of a heavy solid with a gyroscope. Vestnik Moscow Univ., Ser. Mat. i Mekh., №3, 1963.
2. Aminov, M. Sh., Some problems of the motion and stability of a solid of variable mass. Trudy Kazan. Aviats. Inst., №48, 1959.
3. Kirgizbaev, Zh., On the stability of permanent rotations of a variable-mass body with a gyroscope in a Newtonian force field. Trudy Univ. Druzhby Narodov im. Patrice Lumumba, Vol. 27, 1968.
4. Chetaev, N. G., On sufficient conditions for the stability of the rotational motion of a projectile. PMM Vol. 7, №2, 1943.
5. Rumiantsev, V. V., On the stability of motion relative to part of the variables. Vestnik Moscow Univ., Ser. Mat. Mekh., Astron., Fiz., Khimii, №4, 1957.

Translated by M. D. F.

KINEMATIC INTERPRETATION OF THE MOTION OF A BODY IN THE HESS' SOLUTION

PMM Vol. 34, №3, 1970, pp. 567-570

A. M. KOVALEV
(Donetsk)

(Received December 31, 1968)

Kinematic interpretation of the motion of a body is based on kinematic equations put forward by Kharlamov [1]. The moving angular velocity hodograph was considered in our earlier paper [2] in which we classified all the characteristic forms of the moving hodograph. In the present paper we shall consider the stationary hodograph in all these cases and give a geometric picture of the motion of a body.

1. The motion of the body can be described as slipless rolling of the moving axoid of angular velocity vector on the stationary axoid. The moving hodograph in the Hess' solution was already fully studied in [2], and we shall make use of the results of this study and take the same notation.

The moving hodograph lies in the plane $\omega_1 = {}^1/_2\, c\omega_2$; its projection on the plane $\omega_1 = 0$ is the curve s the equations of which in polar coordinates ρ and φ ($\omega_2 = \rho\cos\varphi$, $\omega_3 = \rho\sin\varphi$) have the form

$$\rho\rho^\cdot = \sqrt{f(\rho)}, \qquad \rho^2\varphi^\cdot = -\rho^3\cos\varphi + k$$

$$\left(f(\rho) = \rho^2\left[1 - \left(\frac{\rho^2}{c} - h\right)^2\right] - k^2\right) \qquad (1.1)$$

the dot superscript denotes differentiation with respect to the dimensionless time τ.

From the first equation of (1.1) we find that ρ is an elliptical function of the time τ with the period

$$T = 2\int_{\rho_1}^{\rho_2} \frac{\rho d\rho}{\sqrt{f(\rho)}}$$

Considering the curve s for various values of parameters [2] we can distinguish four characteristic cases:

1°. The curve s has a limiting closed trajectory S_* and does not pass through a singular point.

2°. The curve s has a limiting closed trajectory S_* and passes through a singular point.

3°. There is no closed limiting trajectory S_* for the curve s, it belongs to the type F_1, apart from that section which crosses the line L ; this section belongs to the type F_2.

4°. There is no closed limiting trajectory S_* for the curve s, it belongs to the type F_1.

Let us consider the motion of the body in each of these cases.

2. The stationary hodograph is constructed on the basis of kinematic equations presented by Kharlamov in [1]

$$\omega_\zeta(\sigma) = \omega_1(\sigma)\nu_1(\sigma) + \omega_2(\sigma)\nu_2(\sigma) + \omega_3(\sigma)\nu_3(\sigma)$$

$$\omega_\rho^2(\sigma) = \omega_1^2(\sigma) + \omega_2^2(\sigma) + \omega_3^2(\sigma) - \omega_\zeta^2$$

$$\omega_\rho^2 \frac{d\alpha}{d\sigma} = \begin{vmatrix} \nu_1(\sigma) & \nu_2(\sigma) & \nu_3(\sigma) \\ \omega_1(\sigma) & \omega_2(\sigma) & \omega_3(\sigma) \\ \dfrac{d\omega_1}{d\sigma} & \dfrac{d\omega_2}{d\sigma} & \dfrac{d\omega_3}{d\sigma} \end{vmatrix}$$

Taking τ as the independent variable of σ, we have

$$\omega_\zeta = {}^1/_2\,\rho\cos\varphi\,(\rho^2 - ch) + k,\quad \omega_\rho^2 = \rho^2\,({}^1/_4 c^2\cos^2\varphi + 1) - \omega_\zeta^2 \tag{2.1}$$

$$\alpha^{\cdot} = \frac{1}{\omega_\rho^2}(k - \rho^3\cos\varphi)\left(\frac{\rho^2}{c} - h - \frac{c}{2}\nu_1\right) \qquad \left(\nu_1 = \frac{k}{\rho}\cos\varphi \pm \frac{1}{\rho}\sin\varphi\,\sqrt{f(\rho)}\right) \tag{2.2}$$

Equations (1.1) define the dependence of ρ and φ on time τ . The sign in front of the root in the expression for ν_1 depends on the choice of initial conditions. Let us take them in such a way that the sign is positive.

Equations (2.1) define the meridian of the surface of revolution on which lies our stationary hodograph. The meridian is bounded by the straight lines

$$\omega_\rho = \frac{1}{\sqrt{\rho_1^2 - k^2}}(k\omega_\zeta - \rho_1^2), \qquad \omega_\rho = \frac{1}{\sqrt{\rho_2^2 - k^2}}(k\omega_\zeta - \rho_2^2) \tag{2.3}$$

In the Cases 1° and 2°, the meridian tends to the limiting closed curve for $\tau \to \infty$ and in the cases 3° and 4° such limiting curve for the meridian does not exist.

Let us explain the conditions under which ω_ρ vanishes. Using Eq. (2.1) we shall represent ω_ρ^2 as a function of $\cos\varphi$, and then find the values of φ for which $\omega_\rho = 0$. We obtain the equation

$${}^1/_4 c^2\rho^2\left[1 - \left(\frac{\rho^2}{c} - h\right)^2\right]\cos^2\varphi - ck\rho\left(\frac{\rho^2}{c} - h\right)\cos\varphi + \rho^2 - k^2 = 0$$

This equation has real solutions only for $f(\rho) = 0$, i.e. when ρ assumes the limit value ρ_1 or ρ_2. In this case

$$\varphi = \pm\arccos\left(2\sqrt{\rho^2 - k^2}/ck\right) \tag{2.4}$$

There are, therefore, some points on the moving hodograph for which $\omega_\rho = 0$; the number of such points varies between 4 and 0, depending on whether $2\sqrt{p^2 - k^2} / ck$ is greater or smaller than unity for $\rho = \rho_1$ or ρ_2.

At the instants when φ has the values determined by (2.4) the stationary hodograph passes through the vertical axis.

It can be proved from (2.2) that at the instants when the curve s intersects line L the direction of variation of angle α changes to opposite, i.e. α has then an extremum point.

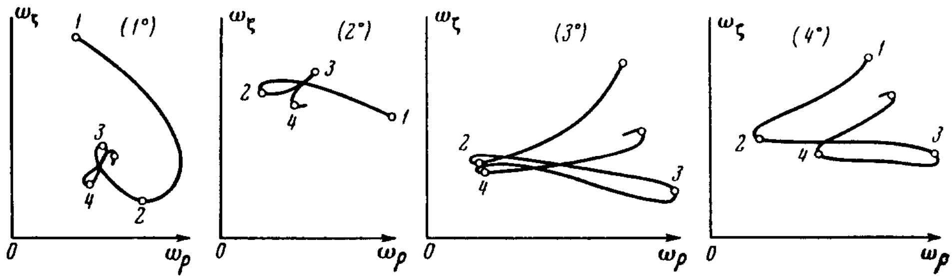

Fig. 1

Figure 1 shows the meridians for Cases 1° to 4°. In the Cases 1°, 2° and 4° the initial point of the curve s was chosen on the external circumference $\rho = \rho_2$ in the domain $G_2(\varphi = 0.1)$, and in the Case 2° in the domain G_1 ($\varphi = -2.99085$).

The following values of the parameters were taken in individual cases:

$$k = 2.0,\ c = 1.5,\ h = 3.0 \qquad (1°,\ 2°)$$

$$k = 0.4,\ c = 0.8,\ h = 0.2 \qquad (3°)$$

$$k = 0.4,\ c = 0.8,\ h = -0.2 \qquad (4°)$$

It must be pointed out that the curve s and the meridian line both rapidly tend to the limiting curves. In the Cases 1° and 2°, even for $\tau > 5T$, the distance between the points of these curves and the corresponding points of limiting curves is less than 0.0001.

For these cases, the stationary hodograph is represented in Fig. 2 as curve H.

3. Interpretation of the motion can now proceed as follows. We first construct the moving hodograph, and for this purpose we project the curve s onto plane $\omega_1 = {}^1/_2\, c\omega_2$. We specify the initial values of ρ and φ, i.e. we choose the initial point on the moving hodograph. From (2.1) we obtain ω_ζ and ω_ρ. This defines the position of the moving hodograph on the stationary hodograph at the initial instant. We obtain the graph of motion by combining those points of the moving and the stationary hodographs which correspond to the same instants of time.

Figure 2 shows the position of the hodographs at some instant. The arrows indicate the direction of the further movement of the osculating point of the two hodographs.

In the Cases 1° and 2° (Fig. 2) the body moves clockwise round the vertical. The asymptotic motion (corresponding to the closed limiting trajectory S_*) is the motion which repeats in space with a period $\tau = T$, if the body is rotated by an angle $\Delta\alpha$ round the vertical. For the chosen values of parameters $\Delta\alpha = 0.1398$. Quantities c, h and k can be chosen so that the asymptotic motion is periodic, i.e. $\Delta\alpha = 0$.

In the Case 3° (Fig. 2) the body moves in such a way that the angle α oscillates round the zero position in both directions.

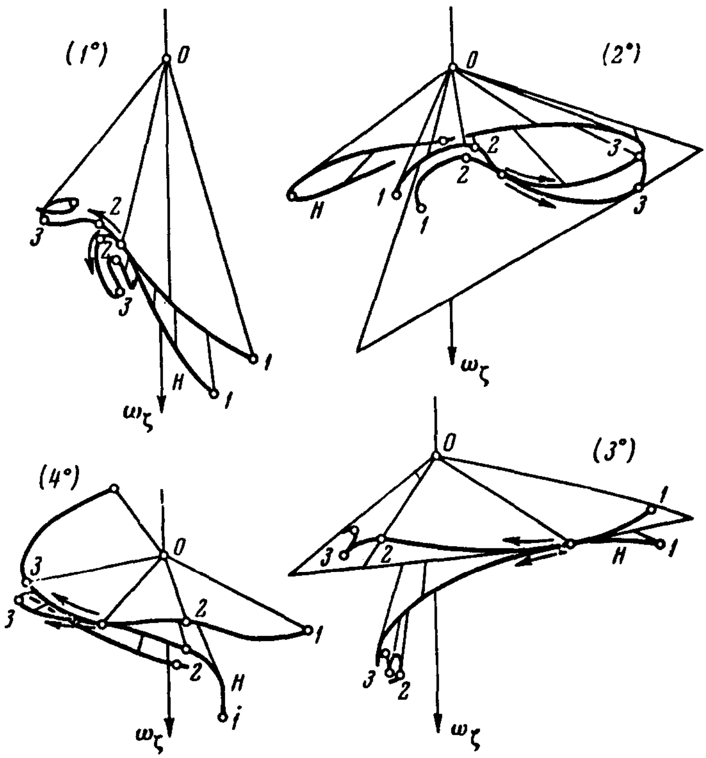

Fig. 2

In the Case 4° (Fig. 2) the angle increases infinitely with the time. The body moves anticlockwise round the vertical.

4. Let us consider the solutions in which $\rho = \text{const}$. The set of equations (1.1) has two such solutions: $\rho = \rho_1$ and $\rho = \rho_2$. In order to satisfy the kinematic equations by these solutions, one of the following equalities must hold

$$k^2 c - 2\rho^4(\rho^3 / c - h) = 0 \quad \text{or} \quad 2c\, 9h - h^3 + (h^2 + 3)\sqrt{h^2+3}] - 27k^2 = 0$$

The meaning of this condition is that $\rho_1 = \rho_2 = \rho_0$, i.e. $f(\rho_0) = 0$, while for all other values of ρ we have $f(\rho) < 0$. The domain G has under these conditions degenerated into a circle which represents the curve s. The moving hodograph is an ellipse.

Let us write now the equations of the stationary hodograph:

$$\omega_\zeta = \frac{k^2 c^2}{4\rho_0^3} \cos\varphi + k, \quad \omega_\rho = \frac{kc}{2}\left(\cos\varphi - \frac{k}{\rho_0^3}\right) \qquad (4.1)$$

$$\frac{d\alpha}{d\varphi} = \frac{2\rho_0^4}{kc\,(k - \rho_0^3 \cos\varphi)} \qquad (4.2)$$

The dependence of φ on time τ is defined by the equation

$$\rho_0^2 \varphi^{\cdot} = -\rho_0^3 \cos\varphi + k$$

When $\cos\varphi$ is eliminated in (4.1), we find that the meridian is a rectilinear section

$$\omega_\zeta = \frac{kc}{2\rho_0^3}\ \omega_\rho + \frac{k^3c^2}{4\rho_0^6} + k$$

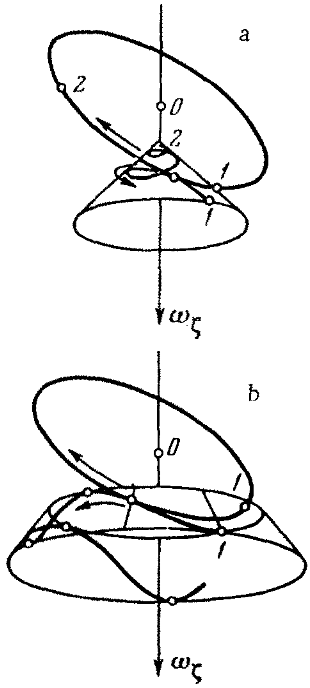

Fig. 3

Further investigation is carried out for two variants.

1. The case of $k \leqslant \rho_0^3$. A point determined by $\varphi_0 = \arccos k / \rho_0^3$ exists on the moving hodograph and this point is reached when $\tau \to \infty$. Equation (4.2) makes it clear that $\alpha \to \infty$ when $\varphi \to \varphi_0$. The stationary hodograph lies on a cone; it is a finite length curve, winding an infinite number of times on the axis ω_ζ.

In this case, the body moves anticlockwise round the vertical and when $\tau \to \infty$ the motion tends to become uniform rotation round the vertical axis with an angular velocity $\omega = {}^1/_4 k^3c^2 / \rho_0^6 + k$. The position of the hodograph is shown in Fig. 3, a (initial point has been taken in domain G_2).

2. The case of $k > \rho_0^3$. When $\tau \to \infty$ we have $\varphi \to \infty$ and $\alpha \to \infty$. The variable point travels round the moving hodograph in a period

$$\tau = T_1 = \frac{2\rho_0^4}{kc}\int_0^{2\pi}\frac{d\varphi}{k - \rho_0^3\cos\varphi}$$

The curve of the stationary hodograph lies on a part of the cone bounded by two planes

$$\omega_\zeta = k + {}^1/_4 k^2c^2 / \rho_0^3,\quad \omega_\zeta = k - {}^1/_4 k^2c^2 / \rho_0^3$$

The stationary hodograph needs to be constructed only for time interval $[0, T_1]$ because its subsequent part which corresponds to the interval $[T_1, 2T_1]$ can be obtained from the first part by simply turning it by the angle $\alpha_0 = 4\rho_0^4\pi / kc\sqrt{k - \rho_0^6}$.

At the initial instant ($\tau = 0$) we combine that point of the moving hodograph which corresponds to $\varphi = 0$ with the point of the stationary hodograph lying on the lower parallel. At the instant $\tau = kT_1$ ($k = 0, 1, 2 \ldots$) the moving hodograph touches the lower parallel and at the instants $\tau = {}^1/_2(2k + 1)T_1$ it touches the upper parallel. The body is in precessional motion round the vertical (Fig. 3, b).

The author thanks P. V. Kharlamov for guidance and G. V. Mozalevskaia for valuable discussion.

BIBLIOGRAPHY

1. Kharlamov, P. V., Lectures on Dynamics of Solids. Special Course, Part 1. Novosibirsk, Novosibirsk Univ., 1965.
2. Kovalev, A. M., The moving angular velocity hodograph in Hess' solution of the problem of motion of a body with a fixed point. PMM Vol. 32, №6, 1968.

Translated by J. M. S.